POWER SERIES SOLUTION OF THE MODIFIED KDV EQUATION

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Abstract. We use the method of Christ [3] to prove local well-posedness of a modified mKdV equation in $\mathcal{F}L^{s,p}$ spaces.

1. Introduction

The mKdV equation on the torus is

$$\left\{ \begin{array}{c} \partial_t u + \partial_x^3 u + u^2 \partial_x u = 0 \\ u(\cdot, 0) = u_0 \end{array} \right. \tag{1}$$

where $u \in H^s(\mathbb{T})$ is a real-valued function of $(x,t) \in \mathbb{T} \times \mathbb{R}$. If u is a smooth solution of (1) then $\|u(\cdot,t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2(\mathbb{T})}$ for all t, therefore $\widetilde{u}(x,t) = u(x + \frac{1}{2\pi}\|u_0\|^2_{L^2(\mathbb{T})} t, t)$ is a solution of

$$\left\{ \begin{array}{c} \partial_t u + \partial_x^3 u + \left(u^2 - \frac{1}{2\pi}\int_{\mathbb{T}} u^2(x,t)dx\right) \partial_x u = 0 \\ u(\cdot, 0) = u_0 \end{array} \right. \tag{2}$$

Thus, (2) and (1) are essentially equivalent. Using Fourier restriction norm method, Bourgain [1] showed that (2) is locally well-posed when $s \geq 1/2$, with uniformly continuous dependence on the initial data u_0. In [2], he also showed that when $s < 1/2$, the solution map is not C^3. Takaoka and Tsutsumi [10] proved local-wellposedness of (2) when $s > 3/8$. For (1), Kappeler and Topalov [8] used inverse scattering method to show wellposedness when $s \geq 0$ and Christ, Colliander and Tao [4] showed that uniformly continuous dependence on the initial data does not hold when $s < 1/2$. Thus, there is a gap between known local well-posedness results and the space $H^{-1/2}(\mathbb{T})$ suggested by the standard scaling argument.

Recently, Grünrock and Vega [7] showed local well-posedness of the mKdV equation on $\mathbb{R}$ with initial data in

$$\widehat{H^r_s}(\mathbb{R}) := \{f \in \mathcal{D}'(\mathbb{R}) : \|f\|_{\widehat{H^r_s}} := \left\| \langle \cdot \rangle^s \hat{f}(\cdot) \right\|_{L^{r'}} < \infty\},$$

when $2 \geq r > 1$ and $s \geq \frac{1}{2} - \frac{1}{2r}$. (for $r > \frac{4}{3}$, this was obtained by Grünrock [5]). This is an extension of the result of Kenig, Ponce and Vega [9] that local-wellposedness holds in $H^s(\mathbb{R})$ when $s \geq 1/4$. Furthermore, as $\widehat{H^r_s}$ scales like H^σ with $\sigma = s + \frac{1}{2} - \frac{1}{r}$, this result covers spaces that have scaling exponent $-\frac{1}{2}+$.

There is also a related recent work of Grünrock and Herr [6] on the derivative nonlinear Schrödinger equation on $\mathbb{T}$. Both [7] and [6] used a version of Bourgain's method.

2000 *Mathematics Subject Classification.* 35Q53.

In this paper, we will apply the new method of solution of Christ [3] to investigate the local well-posedness of (2) with initial data in

$$\mathcal{F}L^{s,p}(\mathbb{T}) := \{f \in \mathcal{D}'(\mathbb{T}) : \|f\|_{\mathcal{F}L^{s,p}} := \left\| \langle \cdot \rangle^s \hat{f}(\cdot) \right\|_{l^p} < \infty\}.$$

Let $B(0,R)$ be the ball of radius R centered at 0 in $\mathcal{F}L^{s,p}(\mathbb{T})$. Our main result is the following.

Theorem 1.1. *Suppose $s \geq 1/2$, $1 \leq p \leq \infty$ and $p'(s+1/4) > 1$. Let W be the solution map for smooth initial data of (2). Then for any $R > 0$ there is $T > 0$ such that, the solution map W extends to a uniformly continuous map from $B(0,R)$ to $C([0,T], \mathcal{F}L^{s,p}(\mathbb{T}))$.*

We note that the $\mathcal{F}L^{s,p}(\mathbb{T})$ spaces that are covered by Theorem 1.1 have scaling index $\frac{1}{4}+$. The restriction $s \geq 1/2$ is due to the presence of the derivative in the nonlinear term, and is only used to bound the operator S_2 in section 3. The same restriction on s is also required in the work on the derivative nonlinear Schrödinger equation on $\mathbb{T}$ by Grünrock and Herr [6]. We believe, however, that the range of p in Theorem 1.1 is not sharp.

Concerning (1), we have the following.

Corollary 1.2. *Suppose $s \geq 1/2$, $1 \leq p \leq \infty$ and $p'(s+1/4) > 1$. Let $\widetilde{W}$ be the solution map for smooth initial data of (2). Then for any $R > 0$ there is $T > 0$ such that for any $c > 0$, the solution map $\widetilde{W}$ extends to a uniformly continuous map from $B(0,R) \cap \{\varphi : \|\varphi\|_{L^2} = c\} \subset \mathcal{F}L^{s,p}(\mathbb{T})$ to $C([0,T], \mathcal{F}L^{s,p}(\mathbb{T}))$.*

As in [3], the solution map W obtained in Theorem 1.1 gives a weak solution of (2) in the following sense. Let T_N be defined by $T_N u = \left(\chi_{[-N,N]}\widehat{u}\right)^{\vee}$. Let $\mathcal{N}u := \left(u^2 - \frac{1}{2\pi}\int_{\mathbb{T}} u^2(x,t)dx\right)\partial_x u$ be the limit in $C([0,T], \mathcal{D}'(\mathbb{T}))$ of $\mathcal{N}(T_N u)$ as $N \to \infty$, provided it exists.

Proposition 1.3. *Let s and p be as in Theorem 1.1. Let $\varphi \in \mathcal{F}L^{s,p}$ and $u := W\varphi \in C([0,T], \mathcal{F}L^{s,p})$. Then $\mathcal{N}u$ exists and u satisfies (2) in the sense of distribution in $(0,T) \times \mathbb{T}$.*

To prove these results, we will formally expand the solution map into a sum of multilinear operators. These multilinear operators are described in the section 2. Then we will show that if $u(\cdot,0) \in \mathcal{F}L^{s,p}$ then the sum of these operators converges in $\mathcal{F}L^{s,p}$ for small time t, when s and p satisfy the conditions of Theorem 1.1. Furthermore, this gives a weak solution of (2), justifying our formal derivation.

Acknowledgement. I would like to thank my advisor Carlos Kenig for suggesting the topic and helpful conversations. I would also like to thank Axel Grünrock and Sebastian Herr for valuable comments and suggestions.

2. Multilinear operators

We rewrite (2) as a system of ordinary differential equations of the spatial Fourier series of u (see formula (1.9) of [10], and also Lemma 8.16 of [1]):

$$
\begin{aligned}
\frac{d\hat{u}(n,t)}{dt} - in^3\hat{u}(n,t) &= -i\sum_{n_1+n_2+n_3=n} \hat{u}(n_1,t)\hat{u}(n_2,t)n_3\hat{u}(n_3,t) \\
&\quad +i\sum_{n_1} \hat{u}(n_1,t)\hat{u}(-n_1,t)n\hat{u}(n,t) \\
&= \frac{-in}{3}\sum_{n_1+n_2+n_3=n}^{*} \hat{u}(n_1,t)\hat{u}(n_2,t)\hat{u}(n_3,t) \\
&\quad +in\hat{u}(n,t)\hat{u}(-n,t)\hat{u}(n,t),
\end{aligned} \tag{3}
$$

where the star means the sum is taken over the triples satisfying $n_j \neq n,\ j=1,2,3$.

Let $a(n,t) = e^{in^3t}\hat{u}(n,t)$, then $a_n(t)$ satisfy

$$
\frac{da(n,t)}{dt} = -\frac{in}{3}\sum_{n_1+n_2+n_3=n}^{*} e^{i\sigma(n_1,n_2,n_3)t}a(n_1,t)a(n_2,t)a(n_3,t) + ina(n,t)a(-n,t)a(n,t),
$$

where

$$
\sigma(n_1,n_2,n_3) = (n_1+n_2+n_3)^3 - n_1^3 - n_2^3 - n_3^3 = 3(n_1+n_2)(n_2+n_3)(n_3+n_1).
$$

Or, in integral form,

$$
\begin{aligned}
a(n,t) &= a(n,0) - \frac{in}{3}\int_0^t \sum_{n_1+n_2+n_3=n}^{*} e^{i\sigma(n_1,n_2,n_3)s}a(n_1,s)a(n_2,s)a(n_3,s)ds \\
&\quad +in\int_0^t |a(n,s)|^2 a(n,s)ds.
\end{aligned} \tag{4}
$$

We note that the triples in the sum are precisely those with $\sigma(n_1,n_2,n_3) \neq 0$. If, a is sufficiently nice, say $a \in C([0,T], l^1)$ (which is the case if $u \in C([0,T], H^s(\mathbb{T}))$ for large s) then we can exchange the order of the integration and summation to obtain

$$
\begin{aligned}
a(n,t) &= a(n,0) - \frac{in}{3}\sum_{n_1+n_2+n_3=n}^{*} \int_0^t e^{i\sigma(n_1,n_2,n_3)s}a(n_1,s)a(n_2,s)a(n_3,s)ds \\
&\quad +in\int_0^t |a(n,s)|^2 a(n,s)ds.
\end{aligned} \tag{5}
$$

Replacing the $a(n_j,s)$ in the right hand side by their equations obtained using (5), we get

$$
\begin{aligned}
a(n,t) &= a(n,0) - \frac{in}{3}\sum_{n_1+n_2+n_3=n}^{*} a(n_1,0)a(n_2,0)a(n_3,0)\int_0^t e^{i\sigma(n_1,n_2,n_3)s}ds \\
&\quad +in|a(n,0)|^2 a(n,0)\int_0^t ds + \text{ additional terms} \\
&= a(n,0) - \frac{n}{3}\sum_{n_1+n_2+n_3=n}^{*} \frac{a(n_1,0)a(n_2,0)a(n_3,0)}{\sigma(n_1,n_2,n_3)}(e^{i\sigma(n_1,n_2,n_3)t}-1) \\
&\quad +int|a(n,0)|^2 a(n,0) + \text{ additional terms}
\end{aligned} \tag{6}
$$

The additional terms are those which depends not only on $a(m,0)$. An example of the additional terms is

$$-\frac{nn_3}{9}\sum_{n_1+n_2+n_3=n}^{*} a(n_1,0)a(n_2,0)\sum_{m_1+m_2+m_3=n_3}^{*}\int_0^t e^{i\sigma(n_1,n_2,n_3)s}\int_0^s e^{i\sigma(m_1,m_2,m_3)s'}\times$$
$$a(m_1,s')a(m_2,s')a(m_3,s')ds'ds$$

We refer to section 2 of [3] for more detailed description of these additional terms. Then we can again use (5) for each appearance of $a(m,\cdot)$ in the additional terms, and obtain more complicated terms. Continuing this process indefinitely, we get a formal expansion of $a(n,t)$ as a sum of multilinear operators of $a(m,0)$.

We will now describes these operators and then show that their sum converges. Again, we refer to section 3 of [3] for more detailed explanations. Each of our multilinear operators will be associated to a tree, which has the property that each of its node has either zero or three children. We will only consider trees with this property. If a node v of T has three children, they will be denoted by v_1, v_2, v_3. We denote by T^0 the set of non-terminal nodes of T, and T^∞ the set of terminal nodes of T. Clearly, if $|T| = 3k+1$ then $|T^0| = k$ and $|T^\infty| = 2k+1$.

Definition 2.1. Let T be a tree. Then $\mathcal{J}(T)$ is the set of $j \in \mathbb{Z}^T$ such that if $v \in T^0$ then

$$j_v = j_{v_1} + j_{v_2} + j_{v_3},$$

and either $j_{v_i} \neq j_v$ for all i, or $j_{v_1} = -j_{v_2} = j_{v_3} = j_v$.

We will denote by $v(T)$ be the root of T and $j(T) = j(v(T))$. For $j \in \mathcal{J}(T)$ and $v \in T^0$,

$$\sigma(j,v) := \sigma(j(v_1), j(v_2), j(v_3)).$$

Definition 2.2. $\mathcal{R}(T,t) = \{s \in \mathbb{R}_+^{T^0} :$ if $v < w$ then $0 \le s_v \le s_w \le t\}$.

Using these definitions, we can rewrite (6) as

$$a(n,t) = a(n,0) + \sum_{|T|=4}\omega_T \sum_{j\in\mathcal{J}(T), j(T)=n} na(j(v_1),0)a(j(v_2),0)a(j(v_3),0)\int_{\mathcal{R}(T,t)} c(j,v,s)ds$$
$$+\text{additional terms}$$

here $c(j,v,s) = e^{i\sigma(j,v)s}$, and ω_T is a constant with $|\omega_T| \le 1$.

Continuing the replacement process will lead to

$$a(n,t) = a(n,0) + \sum_{|T|<3k+1}\omega_T \sum_{j\in\mathcal{J}(T), j(T)=n}\prod_{u\in T^0} j_u \prod_{v\in T^\infty} a(j_v,0)\int_{\mathcal{R}(T,t)} c(j,s)ds$$
$$+\text{additional terms}$$

where

$$c(j,s) = \prod_{v\in T^0} c(j,v,s)$$

We will show that the series

$$a(n,0) + \sum_{T}\omega_T \sum_{j\in\mathcal{J}(T), j(T)=n}\prod_{u\in T^0} j_u \prod_{v\in T^\infty} a(j_v,0)\int_{\mathcal{R}(T,t)} c(j,s)ds$$

converges in l^p to a weak solution of (2).

3. l^p CONVERGENCE

Definition 3.1. For a tree T, $j \in \mathcal{J}(T)$, let

$$I_T(t,j) = \int_{\mathcal{R}(T,t)} c(j,s)ds,$$

and

$$S_T(t)(a_v)_{v\in T^\infty}(n) = \omega_T \sum_{j\in\mathcal{J}(T), j(T)=n} \prod_{u\in T^0} j_u \prod_{v\in T^\infty} a_v(j_v) I_T(t,j).$$

We first give an estimate for $I_T(t,j)$ which allows us to bound S_T.

Lemma 3.2. *For $0 \le t \le 1$, $|I_T(j,t)| \le (Ct)^{|T^0|/2} \prod_{v\in T^0} \langle \sigma(j,v)\rangle^{-1/2}$.*

Proof. Let v_0 be the root of T. For $v \in T^0$, define the level of v, denoted $l(v)$, to be the length of the unique path connecting v_0 and v. Let O be the set of $v \in T^0$ for which $l(v)$ is odd, and E those v for which $l(v)$ is even.

First we fix the variables s_v with $v \in E$, and take the integration in the variables s_v with $v \in O$. For each $v \in O$, the result of the integration is

$$\frac{1}{\sigma(j,v)} \left(e^{i\sigma(j,v)s_{\tilde{v}}} - e^{i\sigma(j,v)\max\{s_{v(1)},s_{v(2)},s_{v(3)}\}}\right)$$

if $\sigma(j,v) \neq 0$, and

$$s_{\tilde{v}} - \max\{s_{v(1)}, s_{v(2)}, s_{v(3)}\}.$$

if $\sigma(j,v) = 0$. Here $\widetilde{v}$ is the parent of v. Thus, we obtain the factor

$$\prod_{v\in O} \langle \sigma(j,v\rangle^{-1}$$

and an integral in s_v, $v \in E$ where the integrand is bounded by $2^{|O|}$. As the domain of integration in s_v with $v \in E$ has measure less than $t^{|E|}$, we see that

$$|I_T(j,t)| \le 2^{|T^0|} t^{|E|} \prod_{v\in O} \langle \sigma(j,v)\rangle^{-1}.$$

By switching the role of O and E, we get

$$|I_T(j,t)| \le 2^{|T^0|} t^{|O|} \prod_{v\in E} \langle \sigma(j,v)\rangle^{-1}.$$

Combining these two estimates, we obtain the lemma. □

By the previous lemma,

$$|S_T(t)(a_v)_{v\in T^\infty}(n)| \le (Ct)^{|T^0|/2} \sum_{j\in\mathcal{J}(T):j(T)=n} \prod_{u\in T^0} \langle \sigma(j,u)\rangle^{-1/2} |j_u| \prod_{v\in T^\infty} |a_v(j_v)|.$$

Let

$$\widetilde{S}_T(a_v)_{v\in T^\infty}(n) = \sum_{j\in\mathcal{J}(T):j(T)=n} \prod_{u\in T^0} \langle \sigma(j,u)\rangle^{-1/2} |j_u| \prod_{v\in T^\infty} |a_v(j_v)|,$$

and

$$\widetilde{S}(a_1,a_2,a_3)(n)=\sum_{n_1+n_2+n_3=n}^{*}|n|\left\langle\sigma(n_1,n_2,n_3)\right\rangle^{-1/2}\prod_{i=1}^{3}|a_i(n_i)|+n\left|\prod a_i(n)\right|.$$

It is clear that

$$\widetilde{S}_T(a_v)_{v\in T^\infty}=\widetilde{S}(\widetilde{S}_{T_1}(a_v)_{v\in T_1^\infty},\widetilde{S}_{T_2}(a_v)_{v\in T_2^\infty},\widetilde{S}_{T_3}(a_v)_{v\in T_3^\infty}).$$

where T_i is the subtree of T that contains all nodes u such that $u\le v(T)_i$ (recall that $v(T)$ is the root of T). Hence, to bound S_T, it suffices to bound $\widetilde{S}$. For this purpose, we will use the following simple lemma.

Lemma 3.3. *Let S be the multilinear operator defined by*

$$S(a_1,a_2,a_3)(n)=\sum_{n_1+n_2+n_3=n}m(n_1,n_2,n_3)\prod_{j=1}^{3}a_j(n_j),$$

Let $1\le p\le\infty$. Then for any pair of indices $i\neq j\in\{1,2,3\}$,

$$\|S(a_1,a_2,a_3)\|_{l^p}\le\sup_n\|m(n_1,n_2,n_3)\|_{l^{p'}_{i,j}}\prod_{k=1}^{3}\|a_k\|_{l^p}.$$

Proof. By Holder inequality, for any n,

$$|S(a_1,a_2,a_3)(n)|\le\|m(n_1,n_2,n_3)\|_{l^{p'}_{i,j}}\left\|\prod_{k=1}^{3}a_k\right\|_{l^p_{i,j}}\le\sup_n\|m(n_1,n_2,n_3)\|_{l^{p'}_{i,j}}\left\|\prod_{k=1}^{3}a_k\right\|_{l^p_{i,j}}$$

Taking l^p-norm in n we obtain the lemma. □

To show that $\widetilde{S}$ is a bounded multilinear map on $l^{s,p}:=\{a:\langle\cdot\rangle^s a\in l^p\}$, we will show the boundedness of S on l^p where S has kernel

$$m(n_1,n_2,n_3)=\frac{\langle n\rangle^s|n|}{\langle\sigma(n_1,n_2,n_3)\rangle^{1/2}\prod_{k=1}^{3}\langle n_k\rangle^s}\quad\text{where } n=n_1+n_2+n_3.$$

We split S into sum of two operators S_1 and S_2 where S_1 has convolution kernel

$$m_1(n_1,n_2,n_3)=\frac{\langle n\rangle^s|n|}{\prod_{k=1}^{3}\langle n_k\rangle^s\langle n-n_k\rangle^{1/2}}\quad\text{if } n=n_1+n_2+n_3,\ \ n_i\neq n$$

and S_2 has kernel

$$m_2(n_1,n_2,n_3)=n/\langle n\rangle^{2s}\quad\text{if } n_1=-n_2=n_3=n.$$

Clearly, for S_2 to be bounded, we need $s\ge1/2$. It remains to bound S_1, for which we have the following.

Proposition 3.4. *S_1 is bounded from $l^p\times l^p\times l^p$ to l^p when $s\ge1/4$ and $p'(s+\frac14)>1$.*

Proof. In the proof, all the sums are taken over the triples (n_1,n_2,n_3) that satisfy the additional property that $n_i\neq n$, for all $1\le i\le3$. Clearly, we can assume $n>0$. Note that if say $|n_1|\ge5n$ then as $|n_2+n_3|=|n-n_1|\ge4n$, at least one of n_2 and n_3 has absolute value bigger than $2n$. Also, we cannot have $|n_i|\le n/4$ for all i. Thus, up to permutation, there are four cases.

(1) $|n_1|, |n_2|, |n_3| \in [n/4, 5n]$
(2) $|n_1|, |n_2| \in [n/4, 5n]$, $|n_3| \leq n/4$
(3) $|n_1| \in [n/4, 5n]$, $|n_2|, |n_3| \leq n/4$
(4) $|n_1|, |n_2| \geq 2n$

By the previous lemma, it suffices to show that in each of these four regions, for some $i \neq j$ the $l^{p'}_{i,j}$-norm of m is bounded.

Case 1. As $3n = \sum(n-n_i)$ for some index i, say $i=3$, we must have $|n-n_3| \sim n$. Since we also have $|n_1|, |n_2| \gtrsim n$,

$$|m(n_1,n_2,n_3)| \lesssim \frac{\langle n\rangle^{1/2-s}}{\langle n_3\rangle^s |(n-n_1)(n-n_2)|^{1/2}}.$$

We will use the following inequality

$$\left|\frac{1}{n_3(n-n_2)}\right| = \left|\frac{1}{n_1}\left(\frac{1}{n_3} - \frac{1}{n-n_2}\right)\right| \leq \frac{1}{|n_1|}\left(\frac{1}{|n_3|} + \frac{1}{|n-n_2|}\right).$$

(1) If $1/4 \leq s \leq 1/2$: then $\langle n_3\rangle^{p'(1/2-s)} \lesssim \langle n\rangle^{p'(1/2-s)}$, so

$$\begin{aligned}
\|m\|^{p'}_{l^{p'}_{1,2}} &\lesssim \sum_{|n_1|\leq 5n} \frac{\langle n\rangle^{p'(1/2-s)}}{|n-n_1|^{p'/2}} \sum_{|n_2|\leq 5n} \frac{\langle n_3\rangle^{p'(1/2-s)}}{(\langle n_3\rangle |n-n_2|)^{p'/2}} \\
&\lesssim \sum_{|n_1|\leq 5n} \frac{\langle n\rangle^{p'(1/2-s)}}{|n-n_1|^{p'/2}} \sum_{|n_2|\leq 5n} \frac{\langle n\rangle^{p'(1/2-s)}}{|n_1|^{p'/2}} \left(\frac{1}{|n-n_2|^{p'/2}} + \frac{1}{|n-n_1-n_2|^{p'/2}}\right) \\
&\lesssim \sum_{|n_1|\leq 5n} \frac{\langle n\rangle^{p'(1-2s)} A_n}{|(n-n_1)n_1|^{p'/2}} \\
&\lesssim \langle n\rangle^{p'(1-2s)} A_n \sum_{|n_1|\leq 5n} \left(\frac{1}{n}\left(\frac{1}{|n-n_1|} + \frac{1}{|n_1|}\right)\right)^{p'/2} \\
&\lesssim \langle n\rangle^{p'(1/2-2s)} A_n^2.
\end{aligned}$$

where $\sum_{0<j<5n} j^{-p'/2} = A_n$. As

$$A_n \lesssim \begin{cases} n^{1-p'/2} & \text{if } p' < 2 \\ \log\langle n\rangle & \text{if } p' = 2 \\ 1 & \text{if } p' > 2 \end{cases}$$

we easily check that $\langle n\rangle^{(1/2-2s)p'} A_n^2$ is bounded by a constant, under our hypothesis on s and p'.

(2) If $s > 1/2$: then $\langle n - n_2\rangle^{p'(s-1/2)} \lesssim \langle n\rangle^{p'(s-1/2)}$, so

$$\begin{aligned}
\|m\|^{p'}_{l^{p'}_{1,2}} &\lesssim \sum_{|n_1|\le 5n} \frac{\langle n\rangle^{p'(1/2-s)}}{|n-n_1|^{p'/2}} \sum_{|n_2|\le 5n} \frac{\langle n-n_2\rangle^{p'(s-1/2)}}{(\langle n_3\rangle\,|n-n_2|)^{p's}} \\
&\lesssim \sum_{|n_1|\le 5n} \frac{\langle n\rangle^{p'(1/2-s)}}{|n-n_1|^{p'/2}} \sum_{|n_2|\le 5n} \frac{\langle n\rangle^{p'(s-1/2)}}{|n_1|^{p's}} \left(\frac{1}{|n-n_2|^{p's}} + \frac{1}{|n-n_1-n_2|^{p's}}\right) \\
&\lesssim \sum_{|n_1|\le 5n} \frac{B_n}{|n-n_1|^{p'/2}\,|n_1|^{p's}} \\
&\lesssim B_n \sum_{|n_1|\le 5n} |n-n_1|^{p'(s-1/2)} \left(\frac{1}{n}\Big(\frac{1}{|n-n_1|}+\frac{1}{|n_1|}\Big)\right)^{p's} \\
&\lesssim \langle n\rangle^{-p'/2} B_n^2.
\end{aligned}$$

where $B_n = \sum_{0<j<5n} j^{-p's}$. As

$$B_n \lesssim \begin{cases} n^{1-p's} & \text{if } p's < 1 \\ \log\langle n\rangle & \text{if } p's = 1 \\ 1 & \text{if } p's > 1 \end{cases}$$

we easily check that $\langle n\rangle^{-p'/2} B_n^2$ is bounded by a constant, under our hypothesis on s and p'.

Case 2 This case can be treated in exactly the same way as the first case, except when $n_3 = 0$. In the region $n_3 = 0$,

$$\begin{aligned}
\|m\|^{p'}_{l^{p'}_{1,3}} &\lesssim \sum_{n_1} \frac{\langle n\rangle^{p'(1/2-s)}}{|n_1(n-n_1)|^{p'/2}} \le \sum_{n_1} \langle n\rangle^{-p's} \left(\frac{1}{|n_1|^{p'/2}} + \frac{1}{|n-n_1|^{p'/2}}\right) \\
&\lesssim \langle n\rangle^{-p's} A_n \lesssim 1
\end{aligned}$$

Case 3 As $|n_1|, |n-n_2|, |n-n_3| \sim n$,

$$|m(n_1,n_2,n_3)| \lesssim \frac{1}{\langle n_2\rangle^s \langle n_3\rangle^s |n_2+n_3|^{1/2}}.$$

Without loss of generality, we can suppose $|n_3| \ge |n_2|$

(1) If $|n_2| < |n_3|/2$:

$$\begin{aligned}
\|m\|^{p'}_{l^{p'}_{2,3}} &\lesssim \sum_{0\le|n_2|\le n/4} \frac{1}{\langle n_2\rangle^{p's}} \sum_{n/4\ge|n_3|>2n_2} \frac{1}{\langle n_3\rangle^{p'(s+1/2)}} \\
&\lesssim \sum_{0\le|n_2|\le n/4} \frac{1}{\langle n_2\rangle^{p'(2s+1/2)-1}} \\
&\lesssim 1
\end{aligned}$$

if $(s+1/4)p' > 1$.

(2) If $|n_2| \geq |n_3|/2$:

$$\begin{aligned}
\|m\|^{p'}_{l^{p'}_{2,3}} &\lesssim \sum_{|n_3|\leq n/4} \frac{1}{\langle n_3\rangle^{2p's}} \sum_{|n_3|\geq n_2 \geq |n_3|/2} \frac{1}{\langle n_3+n_2\rangle^{p'/2}} \\
&\lesssim \sum_{|n_3|\leq n/4} \frac{1}{\langle n_3\rangle^{2p's}} \max\{\log\langle n_3\rangle, \langle n_3\rangle^{-p'/2+1}\} \\
&\lesssim \sum_{|n_3|\leq n/4} \frac{\log\langle n_3\rangle}{\langle n_3\rangle^{2p's}} + \sum_{|n_3|\leq n/4} \frac{1}{\langle n_3\rangle^{p'(2s+1/2)-1}} \lesssim 1
\end{aligned}$$

as $2p's \geq p'(s+1/4) > 1$.

Case 4 $|n_1|, |n_2| > 2n$: Note that in this case, $|n_1| \sim |n-n_1|$ and $|n_2| \sim |n-n_3|$.

(1) If $|n_3|, |n-n_3| \geq n/2$: we have

$$|m(n_1,n_2,n_3)| \lesssim \frac{\langle n\rangle^{1/2}}{\langle n_1\rangle^{s+1/2}\langle n_2\rangle^{s+1/2}},$$

hence

$$\begin{aligned}
\|m\|^{p'}_{l^{p'}_{1,2}} &\lesssim \langle n\rangle^{p'/2} \sum_{|n_1|,|n_2|>2n} \frac{1}{\langle n_1\rangle^{p'(s+1/2)}\langle n_2\rangle^{p'(s+1/2)}} \\
&\lesssim \frac{\langle n\rangle^{p'/2}}{\langle 2n\rangle^{p'(2s+1)-2}} \lesssim 1.
\end{aligned}$$

(2) If $|n_3| < n/2$: then $|n_1| \sim |n_2|$ and $|n-n_3| \geq n/2$, so

$$|m(n_1,n_2,n_3)| \lesssim \frac{n^{s+1/2}}{\langle n_1\rangle^{2s+1}\langle n_3\rangle^{s}},$$

hence

$$\|m\|^{p'}_{l^{p'}_{1,3}} \lesssim B_n \sum_{|n_1|>2n} \frac{n^{p'(s+1/2)}}{\langle n_1\rangle^{p'(2s+1)}} \lesssim \frac{B_n}{n^{p'(s+1/2)-1}} \lesssim 1$$

(3) If $|n-n_3| < n/2$: then $|n_1| \sim |n_2|$ and $|n_3| \sim n$. Hence,

$$|m(n_1,n_2,n_3)| \lesssim \frac{n}{\langle n_1\rangle^{2s+1}\langle n-n_3\rangle^{1/2}}.$$

Therefore,

$$\begin{aligned}
\|m\|^{p'}_{l^{p'}_{1,3}} &\lesssim \sum_{|n_1|\geq 2n} \sum_{n/2<n_3<3n/2} \frac{n^{p'}}{\langle n_1\rangle^{p'(2s+1)}\langle n-n_3\rangle^{p'/2}} \\
&\lesssim \sum_{|n_1|\geq 2n} \frac{A_n n^{p'}}{\langle n_1\rangle^{p'(2s+1)}} \lesssim \frac{A_n}{n^{2p's-1}} \lesssim 1
\end{aligned}$$

This concludes the proof of the proposition. $\square$

Proof of Theorem 1.1. Let $u_0 \in \mathcal{F}L^{s,p}$ and $a(n) = \widehat{u_0}(n)$. By the previous proposition,

$$\|S_T((a_v)_{v\in T^\infty})\|_{l^{s,p}} \le C^{|T^0|} t^{|T^0|/2} \prod_{v\in T^\infty} \|a_v\|_{l^{s,p}}.$$

Hence, the sum

$$\left\| a(n,0) + \sum_T \sum_{j\in\mathcal{J}(T), j(T)=n} \prod_{u\in T^0} j_u \prod_{v\in T^\infty} a(j_v,0) \int_{\mathcal{R}(T,t)} c(j,s)ds \right\|_{l^{s,p}} \le$$

$$\sum_T \|S_T(a,\dots,a)\|_{l^{s,p}} \le \sum_{k=0}^{\infty} (Ct)^{k/2} \|a\|_{l^{s,p}}^{2k+1} = \frac{\|u_0\|_{\mathcal{F}L^{s,p}}}{1-\sqrt{Ct}\,\|u_0\|_{\mathcal{F}L^{s,p}}^2}. \tag{7}$$

converges for all $t \lesssim \min\{1, \|u_0\|_{\mathcal{F}L^{s,p}}^{-4}\}$. Let $a(n,t)$ denote this sum, and define the solution map $u = Wu_0$ by $\widehat{u}(n,t) = e^{-in^3t}a(n,t)$. It follows from (7) that W is uniformly continuous. It remains to show that W extends the solution maps for smooth initial data.

From the definition of S_T, it is clear that $a(n,t)$ satisfies the equation (5). Let $u_N(0) = \left(\chi_{[-N,N]}\widehat{u_0}\right)^\vee$ and $u_N = W(u_N(0))$. As $\|u_N(\cdot,0)\|_{\mathcal{F}L^{s,p}} \le \|u(\cdot,0)\|_{\mathcal{F}L^{s,p}}$, u_N is defined on the interval where u is defined, and $u_N \to u$ in $C([0,T], \mathcal{F}L^{s,p})$. Since $\widehat{u}_N(\cdot,0)$ is compactly supported, $u_N \in C([0,T_0], \mathcal{F}L^{\sigma,p}) \subset C([0,T_0], \mathcal{F}L^1)$ for some large σ. Here, T_0 depends on σ and N. Thus, if $t \le T_0$, in (5) we can exchange the order of the sum and the integral, therefore u_N satisfies (4). Thus, u_N is a classical solution of (2). Using the bound (7), we can repeat the argument on the interval $[T_0, 2T_0]$, etc., and show that u_N is a classical solution on an interval $[0,T_1]$ where T_1 depends on $\|u_0\|_{\mathcal{F}L^{s,p}}$ only. Thus u is the limit in $C([0,T_1], \mathcal{F}L^{s,p})$ of smooth solutions u_N. □

The proof of Proposition 1.2 is basically the same as that of Proposition 1.4 in [3], hence we obmit it.

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